

AN ASSORTMENT OF NEGATIVELY CURVED ENDS

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ABSTRACT. Motivated by a recent groundbreaking work of Ontaneda, we describe a sizable class of closed manifolds such that the product of each manifold in the class with \mathbb{R} admits a complete metric of bounded negative sectional curvature which is an exponentially warped near one end and has finite volume near the other end.

1. INTRODUCTION

Let V be a finite volume, complete, open Riemannian manifold of sectional curvature K within $[-1, 0)$. Little is known about topology of V . An argument of Gromov [Gro78] extended by Schroeder [BGS85, Appendix 2] implies that V is diffeomorphic to the interior of a compact manifold \bar{V} with boundary; thus any manifold compactification of V is obtained by attaching an h-cobordism to \bar{V} along $\partial\bar{V}$.

A well-known problem is to determine which manifolds occur as the boundary of \bar{V} . Gromov showed that $\partial\bar{V}$ must have zero simplicial volume [Gro82, p.37], and this seems to be the only known obstruction in the case when each component of $\partial\bar{V}$ is aspherical. Other obstructions were found in [BP] when $\partial\bar{V}$ has a non-aspherical component, denoted C ; e.g. $\pi_1(C)$ cannot be an irreducible, higher rank lattice, or a virtually nilpotent group.

Earlier examples of manifolds appearing as components of $\partial\bar{V}$ include generalized graph manifolds [AS92, Buy93], and some circle bundle over real and complex hyperbolic manifolds [Fuj88, Bel12b, Bel12a]. A recent breakthrough of Ontaneda allows to dramatically expand the list.

If a (not necessarily connected) manifold B is diffeomorphic to the boundary of a connected, smooth, compact manifold N , then we say that B *bounds* N , and if N is not specified, we simply say B *bounds*.

Ontaneda's proof starts with a closed manifold B that bounds, and applies relative strict hyperbolization of Charney-Davis [CD95] to realize B as a boundary of a compact manifold whose interior admits a piecewise hyperbolic metric.

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Then under the assumption that the building block in the hyperbolization is “large enough” Ontaneda is able to smooth the metric away from the boundary to a Riemannian metric with K near -1 ; this smoothing argument is technological tour de force. Near the boundary the metric has to be constructed by an ad hoc method depending on B . The following result is implicit in [Ont].

Theorem 1.1. (Ontaneda) *Let B be a closed n -manifold that bounds and*
 (i) *if $n \geq 5$, then any h -cobordism from B to another manifold is a product;*
 (ii) *$\mathbb{R} \times B$ admits a complete metric g of sectional curvature within $[-1, 0)$ such that $(-\infty, 0] \times B$ has finite g -volume, and $g = dr^2 + e^{2r}g_B$ on $[c, \infty) \times B$ for some $c > 0$ and a metric g_B on B .*
Then B bounds a manifold whose interior admits a complete metric of finite volume and sectional curvature in $[-1, 0)$.

Condition (ii) implies that each component of B is aspherical, and hence has torsion-free fundamental group. The Whitehead Torsion Conjecture, which is true for many groups of geometric origin, predicts that all torsion-free groups have zero Whitehead torsion. If the conjecture is true for the fundamental group of each component of B , then (i) holds.

For example (ii) is true if B is any infranilmanifold [BK05], or any 3-dimensional *Sol* manifolds [Pha]. We add to the list as follows.

Theorem 1.2. *Condition (ii) holds for every manifold in the class \mathcal{B} that is defined as the smallest class of closed manifolds of positive dimension such that*
 • *\mathcal{B} contains each infranilmanifold, every circle bundle of type (K), and each closed manifold of $K \leq 0$ with a local Euclidean de Rham factor;*
 • *\mathcal{B} is closed under products, disjoint unions, and products with any compact manifold of $K \leq 0$.*

An orientable circle bundle *has type (K)* if its base is a closed complex hyperbolic n -manifold whose holonomy representation lifts to $U(n, 1)$, and if the Euler class of the bundle equals $-m\frac{\omega}{4\pi}$ for some nonzero integer m , where ω is the Kähler form of the base. For example, every orientable circle bundle over a genus two, orientable, closed surface has type (K).

It is immediate from [BL, BFL] that (i) holds for every manifold in \mathcal{B} , so Theorems 1.1–1.2 imply

Corollary 1.3. *If a manifold B in \mathcal{B} bounds, then B bounds a compact connected manifold whose interior V admits a complete Riemannian metric of finite volume and sectional curvature in $[-1, 0)$.*

Theorem 1.2 is proved by showing that each manifold in \mathcal{B} carries what we call a simultaneously diagonalizable family of metrics g_r such that $g = dr^2 + g_r$

is as in (ii), and the main observation is that simultaneous diagonalizability behaves well under products, and facilitates curvature computations.

Results of this paper can be modified to hold for other curvature conditions such as $K \leq -1$, or $-1 \leq K \leq 0$, and other growth assumptions on the ends, such as infinite volume or $\text{Rad Inj} \rightarrow 0$, which is all left to an interested reader to explore.

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2. CURVATURE FORMULAS

Let us review the curvature formulas for the metric $g = dr^2 + g_r$ on $I \times B$ that were derived in [BW04, Section 6] and corrected in [Bel12b, Appendix C], where I is an open interval, and B is a manifold. The family of metrics (B, g_r) , $r \in I$ is called *simultaneously diagonalizable* if near each point of B there is a basis of vector fields $\{X_i\}$ that is g_r -orthogonal for each r . Set $h_i(r) = \sqrt{g_r(X_i, X_i)}$ and note that $Y_i = X_i/h_i$ form a g_r -orthonormal basis. Denote $g(X, Y)$, $\frac{\partial}{\partial r}$ by $\langle X, Y \rangle$, ∂_r , respectively. If g_r are simultaneously diagonalizable, the components of the curvature tensor of $g = dr^2 + g_r$ are

$$(2.1) \quad \langle R_g(Y_i, Y_j)Y_j, Y_i \rangle = \langle R_{g_r}(Y_i, Y_j)Y_j, Y_i \rangle - \frac{h'_i h'_j}{h_i h_j},$$

$$(2.2) \quad \langle R_g(Y_i, Y_j)Y_l, Y_m \rangle = \langle R_{g_r}(Y_i, Y_j)Y_l, Y_m \rangle \quad \text{if } \{i, j\} \neq \{l, m\},$$

$$(2.3) \quad \langle R_g(Y_i, \partial_r)\partial_r, Y_i \rangle = -\frac{h''_i}{h_i}, \quad \langle R_g(Y_i, \partial_r)\partial_r, Y_j \rangle = 0 \quad \text{if } i \neq j$$

$$(2.4) \quad 2\langle R_g(\partial_r, Y_i)Y_j, Y_k \rangle =$$

$$\langle [Y_i, Y_j], Y_k \rangle \left(\ln \frac{h_k}{h_j} \right)' + \langle [Y_k, Y_i], Y_j \rangle \left(\ln \frac{h_j}{h_k} \right)' + \langle [Y_k, Y_j], Y_i \rangle \left(\ln \frac{h_i^2}{h_j h_k} \right)'$$

The second fundamental form \mathbb{I}_{g_r} of g_r is given by $\mathbb{I}_{g_r}(Y_i, Y_j) = 0$ if $i \neq j$, and $\mathbb{I}_{g_r}(Y_i, Y_i) = -\frac{h'_i}{h_i} \partial_r$ [BW04].

3. PRODUCTS

Given manifolds B_1, B_2 consider the metrics $dr^2 + g_{r,k}$ on $\mathbb{R} \times B_k$ with $k = 1, 2$, and $g = dr^2 + g_r$ on $\mathbb{R} \times B_1 \times B_2$ where $g_r = g_{r,1} + g_{r,2}$. Denote the negative reals by \mathbb{R}_- .

Theorem 3.1. *For $k = 1, 2$ suppose that $g_{r,k}$ is simultaneously diagonalizable, and \mathbb{I}_{g_r} is negative definite for all r . Then*

- (1) $K_g < 0$ if $K_{dr^2+g_{r,1}}, K_{dr^2+g_{r,2}}$ are negative.
- (2) K_g is bounded if $K_{dr^2+g_{r,1}}, K_{dr^2+g_{r,2}}, \mathbb{I}_{g_r}$ are bounded.
- (3) $\text{Vol}(\mathbb{R}_- \times B_1 \times B_2, g)$ is finite if $\text{Vol}(\mathbb{R}_- \times B_1, dr^2 + g_{r,k})$ and $\text{Vol}(B_2, g_{0,3-k})$ are finite for some k .

Remark 3.2. If \mathbb{I}_{g_r} is positive definite, the same result holds with \mathbb{R}_- replaced by the positive reals.

Proof. The product $g_r = g_{r,1} + g_{r,2}$ is simultaneously diagonalizable, so let Y_i be the corresponding basis vectors tangent to one of the factors. For brevity denote $g\langle R_g(A, B)C, D \rangle$ by $(A, B, C, D)_g$. Fix vectors C, D , and write $C = C_1 + C_2$, $D = D_1 + D_2$ where C_k, D_k are tangent to B_k . Fix reals a, b such that $a\partial_r + C, b\partial_r + D$ are orthonormal.

Think of g is the Riemannian submersion metric with base $dr^2 + g_{r,k}$ and fiber $g_{r,3-k}$. Since the submersion metric g is a warped product, its A -tensor vanishes. Hence O'Neill's formula R_g restricted to the normal bundle to the horizontal space equals $R_{dr^2 + g_{r,k}}$, and hence the g -sectional curvature of any horizontal plane is negative under the assumptions of (1) and bounded under the assumptions of (2). Thus if we show that

$$(a\partial_r + C, b\partial_r + D, b\partial_r + D, a\partial_r + C)_g \leq \sum_{k=1}^2 (a\partial_r + C_k, b\partial_r + D_k, b\partial_r + D_k, a\partial_r + C_k)_g$$

then (1) would follow. Now $(a\partial_r + C, b\partial_r + D, b\partial_r + D, a\partial_r + C)_g$ equals

$$2ab(\partial_r, D, \partial_r, C)_g + a^2(\partial_r, D, D, \partial_r)_g + 2a(\partial_r, D, D, C)_g + b^2(C, \partial_r, \partial_r, C)_g + 2b(C, \partial_r, D, C)_g + (C, D, D, C)_g$$

and each $(a\partial_r + C_k, b\partial_r + D_k, b\partial_r + D_k, a\partial_r + C_k)_g$ has a similar decomposition into six summands. The desired inequality is to be established one summand at a time, and in fact, it is an equality except for the last summand.

As $(\partial_r, Y_i, \partial_r, Y_j)_g = 0$ for $i \neq j$ we see that $(\partial_r, D, \partial_r, C)_g = (\partial_r, D_1, \partial_r, C_1)_g + (\partial_r, D_2, \partial_r, C_2)_g$, and similar equalities hold for $(\partial_r, D, D, \partial_r)_g, (C, \partial_r, \partial_r, C)_g$. As $(\partial_r, Y_i, Y_j, Y_k)_g = 0$ unless Y_i, Y_j, Y_k are tangent to the same factor, we deduce $(\partial_r, D, D, C)_g = (\partial_r, D_1, D_1, C_1)_g + (\partial_r, D_2, D_2, C_2)_g$; a similar formula holds for $(C, \partial_r, D, C)_g$.

The last summand is given by the Gauss formula

$$(C, D, D, C)_g = (C, D, D, C)_{g_r} + \langle \mathbb{I}_{g_r}(C, D), \mathbb{I}_{g_r}(C, D) \rangle_g - \langle \mathbb{I}_{g_r}(C, C), \mathbb{I}_{g_r}(D, D) \rangle_g$$

and again a similar formula holds for $(C_k, D_k, D_k, C_k)_g$.

Since g_r is the product metric, $(C_i, D_j, D_k, C_m)_{g_r} = 0$ unless i, j, k, l are all equal, so $(C, D, D, C)_{g_r} = (C_1, D_1, D_1, C_1)_{g_r} + (C_2, D_2, D_2, C_2)_{g_r}$. Let T be the diagonal matrix with ii entry equal to $\sqrt{\frac{|h'_i|}{h_i}}$. Since \mathbb{I}_{g_r} is nonpositive definite, $\langle TA, TB \rangle_g \partial_r = -\mathbb{I}_{g_r}(A, B)$, so

$$\langle \mathbb{I}_{g_r}(C, D), \mathbb{I}_{g_r}(C, D) \rangle_g - \langle \mathbb{I}_{g_r}(C, C), \mathbb{I}_{g_r}(D, D) \rangle_g = \langle TC, TD \rangle_g^2 - \langle TC, TC \rangle_g \langle TD, TD \rangle_g.$$

The same formula holds for C_k, D_k in place of C, D . Combining the above with vanishing of $\langle TC_i, TC_j \rangle_g$, $\langle TD_i, TD_j \rangle_g$, $\langle TD_i, TC_j \rangle_g$ for $i \neq j$ gives

$$\begin{aligned} & (C, D, D, C)_g - (C_1, D_1, D_1, C_1)_g - (C_2, D_2, D_2, C_2)_g = \\ & (\langle TC_1, TD_1 \rangle_g + \langle TC_2, TD_2 \rangle_g)^2 - (|TC_1|_g^2 + |TC_2|_g^2)(|TD_1|_g^2 + |TD_2|_g^2) \leq \\ & (|TC_1|_g|TD_1|_g + |TC_2|_g|TD_2|_g)^2 - (|TC_1|_g^2 + |TC_2|_g^2)(|TD_1|_g^2 + |TD_2|_g^2) = \\ & -(|TC_1|_g|TD_2|_g - |TC_2|_g|TD_1|_g)^2 \leq 0. \end{aligned}$$

which completes the proof of (1).

To prove (2) note that the above inequalities are equalities except in one case where the difference is controlled by $\max_{i,r} \frac{|h'_i|}{h_i}$, which is bounded. Finally $(a\partial_r + C_k, b\partial_r + D_k, b\partial_r + D_k, a\partial_r + C_k)_g$ is the product of two bounded quantities, g -area of the parallelogram and $dr^2 + g_{k,r}$ -sectional curvature of the plane each spanned by $\{a\partial_r + C_k, b\partial_r + D_k\}$, so (2) follows.

Since \mathbb{I}_{g_r} is nonpositive definite, each h_i is nondecreasing, so the identity map $(B_k, g_0) \rightarrow (B_k, g_{r,k})$ with $r < 0$ is 1-Lipschitz, and hence volume nonincreasing. Thus if B_k has finite $g_{0,k}$ -volume, then the $g_{r,k}$ -volume of B_k is uniformly bounded on \mathbb{R}_- . Now (3) follows from the Fubini theorem for the Riemannian submersion metric g with base $dr^2 + g_{r,k}$ and fiber $g_{r,3-k}$. \square

4. NONPOSITIVELY CURVED MANIFOLDS

Here is a souped up version of the fact that $dr^2 + e^{2r}g_B$ has $K < 0$ whenever $K_{g_B} \leq 0$.

Theorem 4.1. *If (B, g_B) is a manifold of bounded nonpositive curvature, then there is a convex, increasing, smooth, positive function h that equals e^r for large r , and such that the sectional curvature $(\mathbb{R} \times B, dr^2 + h^2g_B)$ is negative and bounded below.*

Proof. Any 2-plane tangent to $\mathbb{R} \times B$ is of the form $\text{span}\{X_1, cX_2 + d\partial_r\}$ where each $\{X_1, X_2\}$ are g_B -orthonormal vectors tangent to the B -factor and $c^2 + d^2 = 1$. Let $Y_i := X_i/h$, so that Y_1, Y_2, ∂_r is g -orthonormal, where $g = dr^2 + h^2g_B$. Then $\langle R_g(Y_i, Y_j)\partial_r \rangle_g = 0$ and

$$K_g(Y_1, cY_2 + d\partial_r) = c^2 \left(\frac{K_{g_B}(Y_1, Y_2)}{h^2} - \left(\frac{h'}{h} \right)^2 \right) - d^2 \frac{h''}{h}.$$

which is negative if h', h'' are negative and $K_{g_B} \leq 0$. To make K_g bounded below fix any $\tau > 0$, and let h be an increasing strictly convex function such that $h(r) = e^r + \tau$ for $r < 0$ and $h(r) = e^r$ for $r \geq r_\tau$, which exists is r_τ is

large enough. (e.g. h can be obtained by smoothing the function that equals $e^r + \tau$ for $r < 0$, equals $r + 1 + \tau$ until the line intersects the graph of e^r , and equals e^r after that). Then $K_g(Y_1, cY_2 + d\partial_r)$ is negative and bounded. \square

Corollary 4.2. *Let (B, g_B) be a compact manifold of nonpositive curvature whose universal cover has a Euclidean de Rham factor. Then B has a simultaneously diagonalizable family of metrics g_r such that $g_r = e^{2r} g_B$ for large r , $K_{dr^2+g_r}$ is negative and bounded below, and $\text{Vol}(\mathbb{R}_- \times B, dr^2 + g_r)$ is finite.*

Proof. Let $(\mathbb{R}^s, g_0) \times (X, g_X)$ be a de Rham splitting of the universal Riemannian cover (\tilde{B}, \tilde{g}_B) of (B, g_B) , where g_0 is the standard Euclidean metric, $s > 0$, and X has no Euclidean factors. Consider the metric $\tilde{g} = dr^2 + e^{2r} g_0 + h^2 g_X$ on $\mathbb{R} \times \tilde{B}$ where h is as in Theorem 4.1, and by that theorem $K_{\tilde{g}}$ is negative and bounded below. Since the de Rham decomposition is invariant under isometries, the metric descends to a metric $dr^2 + g_r$ on $\mathbb{R} \times B$ with the same curvature bounds. By compactness B is covered by finitely many “product charts” that lift isometrically to the universal cover to the product of balls in \mathbb{R}^s and X . The product of any such chart with \mathbb{R}_- has finite g -volume, and hence so does $\mathbb{R}_- \times B$. \square

5. INFRANILMANIFOLDS

Let B be an infranilmanifold of nilpotence degree k . The metric on B constructed [BK05] can be modified to satisfy Theorem 1.1(ii), while keeping the sectional curvature almost in $[-k^2, -1]$. This was used but not explained in [Ont], so we supply a proof below.

The metric in [BK05] is of the form $g = dr^2 + g_r$, where g_r is simultaneously diagonalizable, and the only part of (ii) that does not hold for g is that for all $r \geq c$ we have $g_r = e^{2kr} g_B$ where g_B is a fixed metric. To satisfy (ii) we need to “replace” k by 1.

To do so choose a non-increasing function Q of r such that $Q(r) = k$ on $[c, T_1]$ and $Q(r) = 1$ for $r \geq T_2$. By making $T_2 \gg T_1$ we may assume that Q' is uniformly small. Set $q := kc + \int_c^r Q(r) dr$ so that $q(r) = kr$ on $[c, T_1]$ and $q(r) = r + (k-1)c$ for $r \geq T_2$. Set $h := e^q$ and consider the metric $\bar{g} := dr^2 + \bar{g}_r$ such that $\bar{g}_r = g_r$ for $r \leq T_1$, and $\bar{g}_r = h^2 g_B$ for $r \geq c$; the two definitions of \bar{g}_r agree on the overlap $[c, T_1]$. For $r \geq T_2$ we have $\bar{g}_r = e^{2r} e^{2c(k-1)} g_B$, i.e. \bar{g}_r is the e^{2r} multiple of a fixed metric, as desired.

It remains to show that the sectional curvature \bar{g} is almost in $[-k^2, -1]$ for suitable T_1, T_2 , and for this was essentially done in [BK05]. Indeed, if $r \leq T_1$, then $\bar{g}_r = g_r$, so [BK05] applies directly, and if $r \geq T_1$, then the proof in [BK05] shows that up to a small error, the sectional curvature of \bar{g} is expressed via the

quantities $\left(\frac{h'}{h}\right)^2 = Q^2$ and $\frac{h''}{h} = Q' + Q^2$, and the desired claim follows since Q' is small, and $Q \in [1, k]$.

6. CIRCLE BUNDLES OF TYPE (K)

It is unclear which circle bundles over closed manifolds of $K \leq 0$ lie in \mathcal{B} .

Proposition 6.1. *The total space of an oriented circle bundle over closed non-positively curved manifold has a nonpositively curved metric if and only if the bundle has torsion Euler class.*

Proof. The “only if” direction follows because in $\text{CAT}(0)$ groups centralizers virtually split, and since the homotopy class of a fiber circle is central, the bundle becomes trivial in a finite cover, hence its rational Euler class vanishes because finite covers are injective on rational cohomology due to existence of a transfer map. For the “if” direction, note that vanishing of the rational Euler class forces the bundle to be flat Euclidean, see [Miy01] or [OT05]. Thus its total space is a $\pi_1(M)$ -quotient of $\widetilde{M} \times S^1$, where $\pi_1(M)$ acts by deck-transformations on the universal cover \widetilde{M} and via a homomorphism $\pi_1(M) \rightarrow S^1$ on the second factor. The action is isometric, so the total space carries a metric of $K \leq 0$. \square

Remark 6.2. By the above proof, if the bundle has torsion Euler class, the metric of $K \leq 0$ on the total space has a local Euclidean de Rham factor, and hence the total space lies in \mathcal{B} by Section 4.

The case of circle bundles with non-torsion Euler class seems much harder; to date I can only handle type (K) circle bundles, and the proof depends on one of the main results of [Bel12a], which is reviewed below.

Let \mathbf{CH}^n denote the complex hyperbolic space of complex dimension n normalized to have holomorphic sectional curvature -1 . Fix a torsion-free cocompact lattice in $U(n-1, 1)$, where $n \geq 2$, and let $\hat{\Gamma}$ be the image of the lattice under the inclusion $U(n-1, 1) \hookrightarrow U(n, 1)$. Let Γ denote the projection of $\hat{\Gamma}$ in $PU(n, 1)$, which acts on \mathbf{CH}^n holomorphic isometries. The kernel of the projectivization $U(n, 1) \rightarrow PU(n, 1)$ consists of scalar matrices, which is an n -torus, so since $\hat{\Gamma}$ is discrete and torsion free, the projection $\hat{\Gamma} \rightarrow \Gamma$ is an isomorphism.

Then Γ acts freely on \mathbf{CH}^n , and it stabilizes a subspace $\mathbf{CH}^{n-1} \subset \mathbf{CH}^n$ on which it acts with compact quotient $M := \mathbf{CH}^{n-1}/\Gamma$; thus M is a compact, totally geodesic, embedded submanifold of $\bar{M} := \mathbf{CH}^n/\Gamma$. The metric on \mathbf{CH}^n can be written in cylindrical coordinates about \mathbf{CH}^{n-1} as $dr^2 + g_r$ for $g_r = \sinh^2(r)d\phi^2 + \cosh^2\left(\frac{r}{2}\right)\mathbf{k}^{n-1}$ where $r \geq 0$ and \mathbf{k}^{n-1} is the standard metric

on \mathbf{CH}^{n-1} , see [Bel12a, Section 3]. Let F denote the unit normal bundle to \mathbf{CH}^{n-1} in \mathbf{CH}^n , and set $\bar{F} := F/\Gamma$. The proof of [Bel12a, Theorem 1.1(iii)] yields a complete metric on $\mathbb{R} \times F$ of the form $dr^2 + v^2 d\phi^2 + h^2 \mathbf{k}^{n-1}$. The metric has sectional curvature in $[a, 0)$ for some negative constant a , and given $\varepsilon > 0$ can be chosen to satisfy $v = \sinh(r)$ and $h = \cosh(\frac{r}{2})$ for $r \geq \varepsilon$. The metric is invariant under the stabilizer of \mathbf{CH}^{n-1} in the isometry group of \mathbf{CH}^n , and hence it descends to a complete metric on $\mathbb{R} \times \bar{F}$, which we denote $g_{v,h}$. By construction in [Bel12a] the portion $(-\infty, 0] \times \bar{F}$ has finite $g_{v,h}$ -volume.

The subgroup $U(n-1, 1)$ of $U(n, 1)$ commutes with the copy of $U(1)$ where $U(n-1, 1) \times U(1)$ is embedded into $U(n, 1)$ via the map

$$(A, B) \rightarrow \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix},$$

and hence their images in $PU(n, 1)$ also commute. It follows that there is an isometric $U(1)$ -action on \mathbf{CH}^n by rotation about \mathbf{CH}^{n-1} , and the corresponding action on $(\mathbb{R} \times \bar{F}, g_{v,h})$ is also isometric and free, where $U(1)$ -orbits are the fibers of the normal circle bundles $\{r\} \times \bar{F} \rightarrow M$.

Let \bar{F}_m denote the quotient of \bar{F} by $\mathbb{Z}_m \leq U(1)$; this is a principal circle bundle over M whose Euler class is the m th multiple of the Euler class of \bar{F} . Let $\bar{g}_{v,h,m}$ denote the metric on $\mathbb{R} \times \bar{F}_m$ that is descended from $(\mathbb{R} \times \bar{F}, g_{v,h})$; of course, $\bar{g}_{v,h,m}$ and $g_{v,h}$ are locally isometric.

As is explained e.g. in [Bel12a, Lemma 13.1] the first Chern class of normal bundle of M in \mathbf{CH}^n/Γ is represented $-\frac{\omega}{4\pi}$ where ω is the Kähler form of the complex hyperbolic metric on \mathbf{CH}^{n-1}/Γ . Thus the Euler class of the circle bundle $\bar{F}_m \rightarrow M$ equals $-m\frac{\omega}{4\pi}$; we refer to such circle bundles as the type (K).

Example 6.3. If $n = 1$, then M is a closed orientable surface of negative Euler characteristic, and according to [GKL01, Corollary 2.3.4] $-\frac{\omega}{4\pi}$ integrated over M equals $\pm\chi(M)/2$; thus if M has genus 2, then every orientable circle bundle over M has type (K).

The metric $(\mathbb{R} \times \bar{F}_m, g_{v,h,m})$ is simultaneously diagonalizable on the fiber, and it satisfies all properties needed in (ii) except that it is not equal to $d^2 + e^{2r} g_{B_m}$ for large r . In what follows we modify $g_{v,h,m}$ for large r to make (ii) hold. The idea is similar to that of Section 5 but details are more involved. Locally

$$\bar{g}_{v,h,m} = dr^2 + v^2 d\phi^2 + h^2 \mathbf{k}^{n-1}$$

and for $r \geq \varepsilon$ we have $v = \sinh(r)$ and $h = \cosh(\frac{r}{2})$, so the metric is complex hyperbolic, and its sectional curvature is within $[-1, -\frac{1}{4}]$. Small C^2 change of the warping functions affect the curvature only slightly, so as a first step we change v, h so that $v = \frac{e^r}{2}$ and $h = \frac{1}{2}e^{\frac{r}{2}}$ for $r \geq T_0$ provided T_0 is large

enough. Next we wish to change h to $\frac{e^r}{2}$ for large r while keeping curvature negative. To this end let Q be a smooth non-decreasing function such that $Q = \frac{1}{2}$ on $[T_0, T_1]$ and $Q = 1$ for $r \geq T_2$. Let $q(r) := \frac{T_0}{2} + \int_{T_0}^r Q(r)dr$. Setting $h := \frac{e^q}{2}$ defines a metric, which we denote $g_{v,h,m}$, that agrees the previously defined metric for $r \leq T_0$, while if $r \geq T_2$, then $g_{v,h,m} - dr^2$ is the e^{2r} multiple of the fixed metric as desired for (ii).

It remains to show that the curvature remains negatively pinched for $r \geq T_0$. Set $s := \frac{v}{h^2}$; as $h \geq \frac{1}{2}e^{\frac{r}{2}}$ we get $0 < s \leq 2$ for $r \geq T_0$. The sectional curvature of any metric of the form $g_{v,h,m}$ was computed in [Bel12a, Section 9] in terms of h and v , and below we review the results of this computation. Let $\{C, D\}$ denote an orthonormal basis in a two-plane with

$$C = c_0\partial_r + c_1Y_1 + c_2Y_2 + c_3Y_3, \quad D = d_1Y_1 + d_2Y_2,$$

where $c_i, d_j \in \mathbb{R}$ and $\{\partial_r, Y_1, Y_2, Y_3\}$ are orthonormal, and Y_1 is tangent to the circle fiber. The sectional curvature of the plane is then given by

$$(6.4) \quad (d_1c_2 - d_2c_1)^2 K(Y_2, Y_1) + d_1^2 c_3^2 K(Y_3, Y_1) + d_1^2 c_0^2 K(\partial_r, Y_1) + d_2^2 c_0^2 K(\partial_r, Y_2) + d_2^2 c_3^2 K(Y_3, Y_2) + 3d_1d_2c_0c_3 \langle R(\partial_r, Y_1)Y_2, Y_3 \rangle.$$

where

$$\begin{aligned} K(Y_2, Y_1) &= K(Y_3, Y_1) = \frac{s^2}{16} - \frac{v'}{v} \frac{h'}{h} \leq \frac{1}{4} - Q \leq -\frac{1}{4}, \\ K(Y_3, Y_2) &= -\frac{1}{4h^2} - \frac{3}{h^2}c_{23}^2 - 3c_{23}^2 \frac{s^2}{4} - \left(\frac{h'}{h}\right)^2 < -3c_{23}^2 \frac{s^2}{4}, \\ K(\partial_r, Y_1) &= -\frac{v''}{v} = -1, \quad K(\partial_r, Y_2) = -\frac{h''}{h} = -Q' - Q^2, \\ \langle R(\partial_r, Y_1)Y_2, Y_3 \rangle &= -c_{23} \frac{v}{h^2} \left(\frac{v'}{v} - \frac{h'}{h}\right) = -c_{23}s(1 - Q). \end{aligned}$$

where c_{23} is a constant with $|c_{23}| \leq \frac{1}{2}$ defined by $[Y_2, Y_3] = c_{23} \frac{v}{h^2} Y_1$. Now

$$\begin{aligned} d_1^2 c_0^2 K(\partial_r, Y_1) + d_2^2 c_3^2 K(Y_3, Y_2) &< -|d_1c_0|^2 - |d_2c_3|^2 \frac{3c_{23}^2 s^2}{4} = \\ &- \left(|d_1c_0| - |d_2c_3| \frac{\sqrt{3}|c_{23}|s}{2} \right)^2 - |d_1c_0d_2c_3c_{23}|\sqrt{3}s, \end{aligned}$$

while $0 \leq 1 - Q \leq \frac{1}{2}$ implies

$$3d_1d_2c_0c_3 \langle R(\partial_r, Y_1)Y_2, Y_3 \rangle \leq |d_1d_2c_0c_3c_{23}| \frac{3}{2}s$$

so these three terms of (6.4) add up to a nonpositive number. Since the other three terms are also nonpositive, $K(C, D) \leq 0$, and if $K(C, D)$ is zero somewhere, then the coefficients for the latter three terms vanish, and in particular, $d_1c_3 = 0$, so the mixed term $3d_1d_2c_0c_3 \langle R(\partial_r, Y_1)Y_2, Y_3 \rangle$ is not present in the

sum, and elementary linear algebra (as in [Bel12a, Remark 9.6]) leads to a contradiction with $K(C, D) = 0$. Thus $K(C, D) < 0$ for $r \geq T_0$, which completes the proof that every type (K) circle bundle lies in \mathcal{B} .

Remark 6.5. One may hope to generalize the above argument to any circle bundle over a closed manifold of $K \leq 0$ by fixing a connection on the circle bundle, scaling the fiber by v , the horizontal space by h , and trying to find v , h , and the connection to satisfy (ii); I was unable to make this work.

7. PROOF OF THEOREM 1.2

That \mathcal{B} contains each infranilmanifold, every circle bundle of type (K), and each closed manifold of $K \leq 0$ with a local Euclidean de Rham factor is proved in Sections 4–6 and in each case the metric on the the fiber B is simultaneously diagonalizable. Hence Section 3 implies that \mathcal{B} is closed under products, and obviously it is closed under disjoint unions. Finally, if B is a manifold in \mathcal{B} , then inductively it comes with a family g_r of simultaneously diagonalizable metrics such that $dr^2 + g_r$ is as in (ii). If (M, g_M) is a closed manifold of $K \leq 0$, and h is as in Theorem 4.1, then $g_r + h^2 g_M$ is simultaneously diagonalizable, and $dr^2 + g_r + h^2 g_M$ is as in (ii).

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